Solving Equations in Free Lattices

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Let S be a finite set of lattice equations in the variables $\{x_1, \ldots, x_k\}$. A *solution* of S in a free lattice $\mathbf{FL}(X)$ is a k–tuple $(a_1, \ldots, a_k)$ of elements of $\mathbf{FL}(X)$ which satisfies the equations of S. If $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are both solutions we say $(a_1, \ldots, a_k)$ is more general than $(b_1, \ldots, b_k)$, and write $(a_1, \ldots, a_k) \preceq (b_1, \ldots, b_k)$, if there is an endomorphism of $\mathbf{FL}(X)$ mapping a_i to b_i for each i. This makes the set of all solutions of S into a quasiordered set, denoted $\mathbf{Q}(S)$.

Let $\equiv$ be the natural equivalence relation associated with the quasiorder. Unification theory is interested the nature of $\mathbf{Q}(S)$. If every element of $\mathbf{Q}(S)/{\equiv}$ has some minimal element below it, then the *unification type* of S is *unitary*, *finitary*, or *infinitary* depending on the number of minimal elements. In all other cases its type is *nullary*. We show that the type of the solutions of S in $\mathbf{FL}(X)$ does not depend on $|X|$ as long as $|X| \geq k$. We give several examples, one of which shows that there is a set of equations S which has nullary unification.